STEADY SUPERSONIC FLOW OF A VIBRATIONALLY EXCITED GAS AROUND THIN BODIES

A. N. Bogdanov and V. A. Kulikovskii UDC 533.6.01

The usual approximate method of treating the flow around a body in classical gas dynamics is based on the thin-body approximation where the relative thickness δ of the body is small [1-6]. Steady supersonic flow of a relaxing gas around thin bodies was considered in [7, 8], where the flow unperturbed by the body was assumed to be in equilibrium and where the flow parameters were constants. The presence of a thin body in the flow then leads to a slight departure from equilibrium, which can be treated as a small perturbation from the equilibrium state.

In the present paper we consider thin plane bodies and thin bodies of revolution in the steady supersonic flow of a vibrationally excited gas which relaxes to the equilibrium state downstream of the body. The energy produced when the vibrationally excited molecules relax brakes the supersonic flow and this can lead to "thermal crises" (a breakdown of the steady flow state) [9]. A "thermal crisis" can be avoided by limiting the initial relative departure from equilibrium. Hence in addition to δ there exists another small parameter in the problem: the relative departure from equilibrium ε. The solution of the problem is represented as an asymptotic expansion in the two small parameters.

1. The plane or axisymmetric flow of a vibrationally relaxing gas will be described by the system of equations

$$\begin{gathered}(\rho u)_x + (\rho v)_y + \nu\rho v/y = 0,\ \rho(uu_x + vu_y) + p_x = 0,\\ \rho(uv_x + vv_y) + p_y = 0,\\ up_x + vp_y - a^2(u\rho_x + v\rho_y) = -\rho(\gamma - 1)(ue_{kx} + ve_{ky}),\\ ue_{kx} + ve_{ky} = \omega\left(e_k^* - e_k\right),\end{gathered} \tag{1.1}$$

where x and y are the spatial coordinates; $\nu = 0$ and 1 for plane and axisymmetric flow, respectively; ρ, p, and a are the density, pressure, and frozen speed of sound; u and v are the components of the gas velocity along the x and y axes; γ is the adiabatic index; ω is the reciprocal of the relaxation time of the vibrational degrees of freedom; e_k and e_k^* are the energy of the vibrational degrees of freedom and its equilibrium value; the subscripts x and y denote differentiation with respect to the corresponding coordinate.

We assume the following relations for e_k^* and ω [10]

$$\omega = k_1 p \exp\left(-k_2 T^{-1/3}\right),\quad e_k^* = \theta_k R/(\exp(\theta_k/T) - 1).$$

Here T is the transitional temperature; θ_k is the vibrational temperature; R is the gas constant; k_1 and k_2 are positive constants dependent on the properties of the gas. Numerical values of the k_i are given in [10].

We assume that the unperturbed gas flow far from the body is one-dimensional steady supersonic flow along the x axis with velocity u_0, density ρ_0, and pressure p_0 at x = 0. We assume that the axis of the body coincides with the x axis and that the nose of the body is at x = 0. We transform to dimensionless quantities in (1.1) by putting

$$\begin{gathered}\rho = \rho_0\bar{\rho},\ u = u_0\bar{u},\ v = u_0\bar{v},\ p = \gamma p_0\bar{p},\\ e_k = a_0^2\bar{e}_k,\quad e_k^* = a_0^2\bar{e}_k^*,\quad \omega = \bar{\omega}u_0/L,\quad x = L\bar{x},\quad y = L\bar{y}\end{gathered}$$

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(L is the length scale of the thin body). We assume that the length of the body is finite with the end of the body at $\bar{x} = 1$.

At x = 0 the gas is assumed to have a given departure from equilibrium $e_{k0} - \bar{e}_{k0}^* > 0$. This quantity varies with position because heat is produced from the relaxation of the vibrational degrees of freedom toward the equilibrium state. The parameters of the unperturbed flow are determined as functions of $\bar{x}$ by solving a system of equations obtained from (1.1) by putting

$$\bar{v} = 0, \quad \bar{p}_{\bar{y}} = \bar{\rho}_{\bar{y}} = \bar{u}_{\bar{y}} = \bar{v}_{\bar{y}} = \bar{e}_{k\bar{y}} = 0,$$

which corresponds to the assumption that the unperturbed gas flow is one-dimensional flow along the x axis. Then (a zero superscript denotes the unperturbed flow)

$$\bar{u}^0_{\bar{x}} = \frac{\bar{\omega}\left(\bar{e}_k^{*0} - \bar{e}_k^0\right)(\gamma - 1)}{\bar{u}^0\left(\bar{u}^0 M_0^2(\gamma + 1) - 1 - \gamma M_0^2\right)}. \tag{1.2}$$

For supersonic ($M_0{}^2 > 1$) flow of a vibrationally excited ($\bar{e}_k{}^0 \geq \bar{e}_k{}^{*0}$) gas relaxing to equilibrium downstream, it follows from (1.2) that with increasing $\bar{x}$ the velocity $\bar{u}^0$ drops from 1 to the value u_{min} corresponding to the root of the right-hand side of (1.2). The quantities $\bar{p}^0$ and $\bar{\rho}^0$ increase from $1/\gamma$ and 1, respectively, to their maximum values corresponding to $\bar{u}^0$ given by (1.2). The change in the parameters of the unperturbed flow is accompanied by a decrease in the Mach number M^0, which depends on $\bar{u}^0$. Let $\bar{u}^*$ be the value of $\bar{u}^0$ at which $M^0 = 1$. We have

$$M^{0 2} - 1 = \frac{M_0^2 \bar{u}^{0 2} \bar{\rho}^0}{\gamma \bar{p}^0} - 1 = \frac{(\gamma + 1) M_0^2 \bar{u}^0 - 1 - \gamma M_0^2}{\gamma M_0^2 - \gamma M_0^2 \bar{u}^0 + 1}, \tag{1.3}$$

and therefore $M^0 = 1$ when $M_0{}^2(\gamma + 1)\bar{u}^0 - 1 - \gamma M_0{}^2 = 0$. But then it follows from (1.2) that $\bar{u}_x{}^0 = \infty$ and the steady solution does not exist. This phenomenon is known as a "thermal crisis" [9]. The value $\bar{u}^0 = \bar{u}^*$ corresponding to thermal crises is found from (1.3):

$$\bar{u}^* = \frac{1 + \gamma M_0^2}{(\gamma + 1) M_0^2}. \tag{1.4}$$

The value of $\bar{u}^*$ given by (1.4) varies from unity at $M_0 = 1$ to $\gamma/(\gamma + 1) \approx 0.58$ ($\gamma = 1.4$) when $M_0 \to \infty$. Therefore to avoid a "thermal crisis" the change in the dimensionless velocity must satisfy the inequality $0.58 \leq \bar{u}^* < \bar{u}^0 \leq 1$ over the entire flow field and hence the relative change in the parameters of steady supersonic one-dimensional flow of a vibrationally excited gas is small. Since these changes are determined by the energy released upon relaxation, i.e., by the initial (at $\bar{x} = 0$) departure from equilibrium $\bar{e}_{k0}{}^* - \bar{e}_{k0}$, it is natural to introduce the relative initial departure from equilibrium as a small parameter:

$$\varepsilon = \frac{\bar{e}_{k0} - \bar{e}_{k0}^*}{\gamma \bar{e}_{k0}^{*2} \exp\left(\theta_k/T_0\right)\left(\gamma M_0^2 - 1\right) + \left(M_0^2 - 1\right)/(\gamma - 1)} > 0 \tag{1.5}$$

(the denominator is written in a form convenient for further analysis).

We look for a solution in the form of asymptotic expansions in ε and δ (δ is the usual small parameter in the thin-body theory and is of the order of the ratio of the thickness of the body to its length L):

$$\begin{aligned}
\bar{u} &= 1 + \varepsilon u_{10} + \delta u_{01} + \varepsilon^2 u_{20} + \varepsilon\delta u_{11} + \ldots,\\
\bar{\rho} &= 1 - \varepsilon \rho_{10} + \delta \rho_{01} - \varepsilon^2 \rho_{20} - \varepsilon\delta \rho_{11} + \ldots,\\
\bar{p} &= 1/\gamma - \varepsilon p_{10} + \delta p_{01} - \varepsilon^2 p_{20} - \varepsilon\delta p_{11} + \ldots.
\end{aligned} \tag{1.6}$$

$$\bar{v} = \delta v_{01} - \varepsilon\delta v_{11} + \dots,$$
$$\bar{e}_k = \bar{e}_{k0} + \varepsilon e_{k10} + \delta e_{k01} + \varepsilon^2 e_{k20} + \varepsilon\delta e_{k11} + \dots$$

We note that the omitted terms are in general nonregular (for an example in ordinary gas dynamics see [1]).

We next consider the problem of finding the coefficients of the expansions in (1.6). For the quantities of order ε we have the relations

$$u_{10} = \rho_{10} = p_{10}/\mathrm{M}_0^2 = e_{k10}(\gamma - 1)/(\mathrm{M}_0^2 - 1),$$
$$u_{10} = \exp(-\sigma\bar{x}) - 1, \quad \sigma = \bar{\omega}_0\left(\frac{\gamma(\gamma-1)\bar{e}_{k0}^{*2}\exp(\theta_k/T_0)(\gamma\mathrm{M}_0^2 - 1)}{\mathrm{M}_0^2 - 1} + 1\right). \tag{1.7}$$

The ratio L/σ represents the typical relaxation length of the problem. For the quantities of order ε^2 we have

$$u_{20} = \rho_{20} + u_{10}^2 = p_{20}/\mathrm{M}_0^2 = (\gamma-1)/(\mathrm{M}_0^2 - 1)\,e_{k20} - \frac{1}{2}u_{10}^2(\mathrm{M}_0^2 - 1),$$
$$u_{20} = -(E_1 + 2E_2)\,\bar{x}\exp(-\sigma\bar{x}) + \frac{1 - \exp(-\sigma\bar{x})}{\sigma}(E_1 + E_2(1 + \exp(-\sigma\bar{x}))).$$

Here

$$E_1 = \sigma\left(\gamma\mathrm{M}_0^2 + \frac{\mathrm{M}_0^2}{\mathrm{M}_0^2 - 1} + \frac{k_2}{3T_0^{1/3}}(\gamma\mathrm{M}_0^2 - 1)\right);$$
$$E_2 = \left[\mathrm{M}_0^2\sigma + \frac{1}{2}\bar{\omega}_0\left(-1 + \gamma(\gamma-1)\bar{e}_{k0}^{*2}\exp\frac{\theta_k}{T_0}(\gamma\mathrm{M}_0^2 - 1)^2\left(\gamma\bar{e}_{k0}^{*}\exp\frac{\theta_k}{T_0} - \frac{\theta_k}{2T_0} - 1\right)\right)\right].$$

It will be convenient to introduce a new unknown function Φ such that

$$v_{01} = -\Phi_{\bar{x}\bar{y}}, \quad u_{01} = -\Phi_{\bar{x}\bar{x}}, \quad \rho_{01} = \Phi_{\bar{x}\bar{x}} + \frac{1}{\bar{y}^{\nu}}(\Phi_{\bar{y}}\bar{y}^{\nu})_{\bar{y}},$$
$$p_{01} = \mathrm{M}_0^2\Phi_{\bar{x}\bar{x}}, \quad e_{k01} = \frac{1}{(\gamma-1)}\left((1 - \mathrm{M}_0^2)\Phi_{\bar{x}\bar{x}} + \frac{1}{\bar{y}^{\nu}}(\Phi_{\bar{y}}\bar{y}^{\nu})_{\bar{y}}\right).$$

The equation for Φ can be written in the form

$$(\mathrm{M}_0^2 - 1)\Phi_{\bar{x}\bar{x}\bar{x}} + \bar{\omega}_0\left(\mathrm{M}_0^2 - 1 + \gamma(\gamma-1)(\gamma\mathrm{M}_0^2 - 1)\bar{e}_{k0}^{*2}\exp\frac{\theta_k}{T_0}\right)\Phi_{\bar{x}\bar{x}} -$$
$$-\frac{(\Phi_{\bar{x}\bar{y}}\bar{y}^{\nu})_{\bar{y}}}{\bar{y}^{\nu}} - \bar{\omega}_0\left(1 + \gamma(\gamma-1)\bar{e}_{k0}^{*2}\exp\frac{\theta_k}{T_0}\right)\frac{(\Phi_{\bar{y}}\bar{y}^{\nu})_{\bar{y}}}{\bar{y}^{\nu}} = 0\,.$$

As usual in gas dynamics, here it is convenient to transform to the new independent variables ξ and ζ, where $\xi = \bar{x} - \sqrt{\mathrm{M}_0{}^2 - 1}\,\bar{y}$, $\zeta = \sqrt{\mathrm{M}_0{}^2 - 1}\,\bar{y}$. The equation for Φ takes the form

$$A\Phi_{\xi\xi} + (\mathrm{M}_0^2 - 1)\left[\Phi_{\xi\xi\zeta} + \frac{((\Phi_\xi - \Phi_\zeta)_\xi\zeta^{\nu})_\zeta}{\zeta^{\nu}}\right] + B\left(\frac{((\Phi_\zeta - \Phi_\xi)\zeta^{\nu})_\zeta}{\zeta^{\nu}} - \Phi_{\xi\zeta}\right) = 0, \tag{1.8}$$

where

$$A = \bar{\omega}_0(\gamma-1)^2\gamma\mathrm{M}_0^2\bar{e}_{k0}^{*2}\exp\frac{\theta_k}{T_0}; \quad B = -\bar{\omega}_0\left(1 + \gamma(\gamma-1)\bar{e}_{k0}^{*2}\exp\frac{\theta_k}{T_0}\right)(\mathrm{M}_0^2 - 1).$$

Equation (1.8) can be solved using the Laplace transform with respect to ξ [7, 11]. According to the usual rules of the Laplace transform [12] we have

$$\Phi(\xi,\ \zeta) \to \widetilde{\Phi}(s,\ \zeta),\ \Phi_\xi(\xi,\ \zeta) \to s\widetilde{\Phi}(s,\ \zeta) - \Phi(0,\ \zeta),$$
$$\Phi_{\xi\xi}(\xi,\ \zeta) \to s^2\widetilde{\Phi}(s,\ \zeta) - s\Phi(0,\ \zeta) - \Phi_\xi(0,\ \zeta).$$

We assume homogeneous initial conditions

$$\Phi(0,\ \zeta) = \Phi_\xi(0,\ \zeta) = 0. \tag{1.9}$$

These conditions do not imply that perturbations in the approximation considered here are zero along the first characteristic going out from the nose of the body. Indeed

$$u_{01}(0,\ \zeta),\ p_{01}(0,\ \zeta) \sim \Phi_{\xi\xi}(0,\ \zeta),\ \text{a}\ v_{01}(0,\ \zeta) \sim \Phi_{\xi\zeta}(0,\ \zeta) - \Phi_{\xi\xi}(0,\ \zeta).$$

In place of (1.8) we have

$$\zeta\widetilde{\Phi}_{\zeta\zeta} + (\nu - 2\zeta s)\,\widetilde{\Phi}_\zeta + s\left(-\nu + \frac{As\zeta}{B - \left(\mathrm{M}_0^2 - 1\right)s}\right)\widetilde{\Phi} = 0. \tag{1.10}$$

Hence the problem has been reduced to the ordinary differential equation (1.10).

2. For a plane body we have $\nu = 0$ in (1.10). Then

$$\widetilde{\Phi}_{\zeta\zeta} - 2s\widetilde{\Phi}_\zeta + \frac{As^2}{B - \left(\mathrm{M}_0^2 - 1\right)s}\,\widetilde{\Phi} = 0. \tag{2.1}$$

The general solution of (2.1) is

$$\widetilde{\Phi} = C_1(s)\exp(\lambda_1\zeta) + C_2(s)\exp(\lambda_2\zeta),$$

where

$$\lambda_1 = \frac{1}{2}\left(2s + \sqrt{4s^2 - \frac{4As^2}{B - \left(\mathrm{M}_0^2 - 1\right)s}}\right);\ \lambda_2 = \frac{1}{2}\left(2s - \sqrt{4s^2 - \frac{4As^2}{B - \left(\mathrm{M}_0^2 - 1\right)s}}\right).$$

We expand λ_1 and λ_2 in power series in $1/s$

$$\lambda_1 = s\left(1 + \frac{A}{2\left(\mathrm{M}_0^2 - 1\right)s} + O(1/s^2)\right),\ \ \lambda_2 = s\left(-\frac{A}{2\left(\mathrm{M}_0^2 - 1\right)s} + O(1/s^2)\right).$$

We introduce the notation

$$\Lambda = \frac{A}{2\left(\mathrm{M}_0^2 - 1\right)}, \tag{2.2}$$

which is convenient for further calculation. Then $\lambda_1 = s + \Lambda$, $\lambda_2 = -\Lambda$.

The solution of interest here must be bounded as $\zeta \to \infty$, therefore $C_1(s) = 0$ (the part of the solution involving $C_1(s)$ corresponds to perturbations arriving at the surface of the body from infinity and we assume that there are no perturbations of this kind). Then

$$\widetilde{\Phi} = C_2(s)\exp(-\Lambda\zeta). \tag{2.3}$$

The constant of integration C_2 is to be determined from the boundary condition on the surface of the thin body.

We next consider the inversion of the Laplace transform $\tilde{\Phi}$ given by (2.3). Let the inverse transform of $C_2(s)$ be the function $f_1(\xi)$. Then, returning to the variables $\bar{x}$ and $\bar{y}$, we find

$$\Phi = f_1\left(\bar{x} - \sqrt{M_0^2 - 1}\bar{y}\right)\exp\left(-\Lambda\sqrt{M_0^2 - 1}\bar{y}\right). \tag{2.4}$$

The boundary condition is that the gas cannot penetrate through the surface of the thin body (for $\bar{y} = 0$). If $\bar{y} = \delta Y(\bar{x})$ is the generatrix of the body, then $\bar{v} = \bar{u}\delta Y(\bar{x})$ and hence in the first approximation in δ

$$v_{01} = Y_{\bar{x}}. \tag{2.5}$$

On the other hand

$$v_{01} = -\Phi_{\bar{x}\bar{y}}. \tag{2.6}$$

In the limit $\bar{y} \to 0$ we have from (2.4)

$$\Phi_{\bar{x}\bar{y}} = -\sqrt{M_0^2 - 1}\left(f_1''(\bar{x}) + \Lambda f_1'(\bar{x})\right)$$

(the prime denotes a derivative of the function with respect to its argument). Eliminating v_{01} from (2.5) and (2.6), we obtain an ordinary differential equation for f_1:

$$f_1''(\bar{x}) + \Lambda f_1'(\bar{x}) = \frac{1}{\sqrt{M_0^2 - 1}} Y_{\bar{x}}. \tag{2.7}$$

Below we will need $f_1''(\bar{x})$ in order to determine the drag force on the thin body, hence (2.7) will be considered as a first-order differential equation for f_1'. Its solution is

$$f_1'(\bar{x}) = \exp(-\Lambda\bar{x})\left(f_1'(0) + \int_0^{\bar{x}} \exp(\Lambda\eta)\frac{Y_\eta}{\sqrt{M_0^2 - 1}}\,d\eta\right),$$

and so

$$f_1''(\bar{x}) = \frac{Y_{\bar{x}}}{\sqrt{M_0^2 - 1}} - \Lambda\exp(-\Lambda\bar{x})\left(f_1'(0) + \int_0^{\bar{x}} \exp(\Lambda\eta)\frac{Y_\eta}{\sqrt{M_0^2 - 1}}\,d\eta\right). \tag{2.8}$$

To determine the form of the solution it is still necessary to specify $f_1'(0)$. We consider the conditions in the approach stream to the body. Along the acoustic characteristic going out from the nose of the body (i.e., $\xi = 0$) we must have the relations for a weak shock wave [13]

$$\sqrt{M_0^2 - 1}\,\delta u_{01} = -\delta v_{01},\ \delta p_{01} = -M_0^2\delta u_{01},\quad \delta p_{01} = \delta\rho_{01},\ \delta e_{k01} = 0.$$

In terms of the function Φ these relations are

$$\sqrt{M_0^2 - 1}\,\Phi_{\bar{x}\bar{x}} = -\Phi_{\bar{x}\bar{y}},\ (M_0^2 - 1)\Phi_{\bar{x}\bar{x}} = \Phi_{\bar{y}\bar{y}}. \tag{2.9}$$

Calculating the derivatives of Φ using (2.4), we obtain the following conditions for f_1 corresponding to (2.9);

$$f_1'(0) = 0,\quad 2f_1'(0) + \Lambda f_1(0) = 0,$$

which are consistent with (1.9) ($\Phi_\xi(0, \zeta) = 0$) and justify its use. The expression for $f_1''(\bar{x})$ can now be simplified.

The dimensionless drag force per unit length of the profile is

$$\bar{D}_s = \frac{D_s}{\rho_0 u_0^2 L} = \int_0^L \frac{p - p_0}{\rho_0 u_0^2 L} dy = \frac{\delta}{\gamma M_0^2} \int_0^1 (\gamma \bar{p} - 1) Y_{\bar{x}} d\bar{x},$$

where $\gamma \bar{p} - 1 = \varepsilon \gamma p_{10} + \delta \gamma p_{01} = -\varepsilon \gamma M_0{}^2 u_{10} + \gamma M_0{}^2 \delta \Phi_{\bar{x}\bar{x}} = -\varepsilon \gamma M_0{}^2 u_{10} + \gamma M_0{}^2 \delta f_1''$. Substituting (2.8) for $f_1''(\bar{x})$, we have

$$\bar{D}_s = -\varepsilon\delta \int_0^1 u_{10}(\bar{x}) Y_{\bar{x}} d\bar{x} + \frac{\delta^2}{\sqrt{M_0^2 - 1}} \int_0^1 Y_{\bar{x}} \left[Y_{\bar{x}} - \Lambda \int_0^{\bar{x}} \exp(\Lambda(\eta - \bar{x})) Y_\eta d\eta \right] d\bar{x}. \tag{2.10}$$

This equation can be used to calculate the drag on a thin plane tapered body placed in a steady supersonic flow of a vibrationally excited gas. In the absence of relaxation ($\omega_0 \to 0$) (2.10) gives

$$\bar{D}_{s0} = \frac{\delta^2}{\sqrt{M_0^2 - 1}} \int_0^1 (Y_{\bar{x}})^2 d\bar{x} \tag{2.11}$$

which is a known result in gas dynamics [9].

For a finite body ($Y(1) = 0$) the first term in (2.10) is negative since

$$-\int_0^1 u_{10}(\bar{x}) Y_{\bar{x}} d\bar{x} = -u_{10}(1) Y(1) + \int_0^1 Y(\bar{x}) u_{10\bar{x}} d\bar{x} = \int_0^1 Y(\bar{x}) u_{10\bar{x}} d\bar{x} < 0.$$

This term is of order $\varepsilon\delta$ and for small enough δ it exceeds the other terms in absolute value. Then the thin body will be pulled along by the fluid (negative drag). Setting D_s to zero, we obtain the maximum value δ_{max} for which the body experiences negative drag at $\delta < \delta_{max}$:

$$\delta_{max} = \frac{\varepsilon \sqrt{M_0^2 - 1} \int_0^1 u_{10}(\bar{x}) Y_{\bar{x}} d\bar{x}}{\int_0^1 \left[(Y_{\bar{x}})^2 - \Lambda Y_{\bar{x}} \int_0^{\bar{x}} \exp(\Lambda(\eta - \bar{x})) Y_\eta d\eta \right] d\bar{x}}. \tag{2.12}$$

According to (2.12), $\delta_{max} \sim \varepsilon$. Since the coefficients σ and Λ given by (1.7) and (2.2) are practically independent of M_0 for $M_0 > 3$, and since ε is proportional to $1/M_0{}^2$ for $M_0 > 3$ according to (1.5), we have $\delta_{max} \sim 1/M_0$.

We next consider the calculation of the drag force $\bar{D}_s$ using information [10] on the physical properties of gases. We assume that the shape of the body is specified by the equation $Y = 2(1 - \bar{x})\bar{x}$ and that $\delta = 0.1$. The calculated results for molecular nitrogen and carbon monoxide are given in Tables 1 and 2, respectively, where D_1 is calculated from (2.11) and D_2 and D_3 are calculated from (2.10) (D_2 assuming $\varepsilon = 0$). All values of D_i are multiplied by 10^{-2}. It was assumed that $M_0 = 2$.

3. For an axisymmetric body $\nu = 1$ in (1.10). Then

$$\tilde{\Phi}_{\zeta\zeta} + \left(\frac{1}{\zeta} - 2s\right)\tilde{\Phi}_\zeta + s\left(-\frac{1}{\zeta} + \frac{As}{B - (M_0^2 - 1)s}\right)\tilde{\Phi} = 0.$$

In place of $\tilde{\Phi}$ we introduce the new unknown function $z(s, \zeta)$:

$$\tilde{\Phi}(s, \zeta) = z(s, \zeta) \exp(s\zeta),$$

and in place of the equation for $\tilde{\Phi}$ we obtain an equation for z

TABLE 1

T_k, K	p, Pa	T, K	D_1	D_2	D_3	ε	$\bar{\omega}_0$	Λ
2000	101320	300	0,77	0,77	0,72	0,24	0,067	0,00001
2000	101320	1000	0,77	0,061	0,061	0,052	1914,0	60,0
3000	101320	430	0,77	0,77	—0,21	0,35	2,3	0,004

TABLE 2

T_k, K	p, Pa	T, K	D_1	D_2	D_3	ε	$\bar{\omega}_0$	Λ
2000	101320	300	0,77	0,77	0,47	0,27	16,0	0,004
2000	10132	1000	0,77	0,079	0,079	0,054	4834,0	176,0
2000	101320	250	0,77	0,77	—0,11	0,32	3,3	0,0002

$$z_{\zeta\zeta} + \frac{1}{\zeta} z_\zeta - s^2 \left(1 + \frac{A}{B - (M_0^2 - 1)s}\right) z = 0.$$

With the help of the substitutions

$$z = \frac{\hat{z}}{s\sqrt{1 - \frac{A}{B - (M_0^2 - 1)s}}}, \quad \zeta = \frac{\hat{\zeta}}{s\sqrt{1 - \frac{A}{B - (M_0^2 - 1)s}}}$$

it reduces to Bessel's equation of order zero

$$\hat{z}_{\hat{\zeta}\hat{\zeta}} + \frac{1}{\hat{\zeta}} \hat{z}_{\hat{\zeta}} - \hat{z} = 0. \tag{3.1}$$

The general solution of (3.1) is expressed in terms of the modified Bessel functions I_0 and K_0 of zero order [14]

$$\hat{z} = C_3(s)K_0(\hat{\zeta}) + C_4(s)I_0(\hat{\zeta}).$$

The solution of interest here must be bounded when $\hat{\zeta} \to \infty$, hence $C_4(s) = 0$. Returning to the variable ζ and the function $\tilde{\Phi}$, we obtain

$$\tilde{\Phi} = C_3(s) \frac{K_0\left(s\sqrt{1 - \frac{A}{B - (M_0^2 - 1)s}}\,\zeta\right)}{s\sqrt{1 - \frac{A}{B - (M_0^2 - 1)s}}} \exp(s\zeta).$$

Since we are interested in the drag on the thin body, we consider the behavior of the solution for small $\hat{\zeta}$. It is known [14] that

$$K_0(\eta) \sim -\ln(\eta/2),\ I_0(\eta) \sim 1,\ \eta \to 0. \tag{3.2}$$

Expanding the expression under the square root sign in powers of $1/s$, using the asymptotic forms (3.2), and using the series expansion for the exponential function $\exp(s\zeta) = 1 + s\zeta + \ldots$, we obtain in the limits $\zeta \to 0$

$$\tilde{\Phi} = -C_3(s) \frac{\ln\left(\frac{s + \Lambda}{2}\zeta\right)}{s + \Lambda}.$$

Next consider the inversion of the Laplace transform $\tilde{\Phi}$. The inverse transform of

is the combination of functions [12] $\exp(-\Lambda\xi)\ln\frac{2\xi}{\zeta}$. Let the inverse transform of C_3 be the function $f_2(\xi)$. Then using the multiplication rule for transforms [12] we can write

$$\Phi = \int_0^{\xi} f_2(\eta)\exp(-\Lambda(\xi-\eta))\ln\frac{2(\xi-\eta)}{\zeta}\,d\eta. \tag{3.3}$$

The function f_2 is determined from the boundary condition that the gas cannot penetrate through the surface of the thin body. Let the generatrix of the body be $\bar{y} = \delta Y(\bar{x})$. Then the condition that the streamlines are tangent to the surface implies that $\bar{v} = \bar{u}\delta Y_{\bar{x}}$ and we again obtain (2.5). On the other hand, (2.6) is also satisfied. Eliminating v_{01}, we find

$$\Phi_{\bar{x}\bar{y}} = -Y_{\bar{x}}. \tag{3.4}$$

The integrand of (3.3) has a singularity at $\eta = \xi$, therefore it cannot be differentiated directly. This difficulty can be avoided by introducing the new variable of integration $t = \xi - \eta$. Then $t \in (0, \xi)$, $\eta = \xi - t$, $dt = -d\eta$. In place of (3.3) we will have

$$\Phi = \int_0^{\xi} f_2(\xi - t)\exp(-\Lambda t)\ln\frac{2t}{\zeta}\,dt.$$

Since $\xi \to \bar{x}$ when $\bar{y} \to 0$, we obtain

$$\Phi_{\bar{x}} = \left[f_2(0)\exp(-\Lambda\bar{x}) + \int_0^{\bar{x}} f_2'(\bar{x}-t)\exp(-\Lambda t)\,dt\right]\ln\frac{2}{\sqrt{M_0^2-1}\,\bar{y}} +$$
$$+ f_2(0)\exp(-\Lambda\bar{x})\ln\bar{x} + \int_0^{\bar{x}} f_2'(\bar{x}-t)\exp(-\Lambda t)\ln t\,dt$$

(the prime denotes a derivative of the function with respect to its argument). The value of $f_2(0)$ must be specified in correspondence with the boundary conditions. The relations for a weak shock wave [13] are not satisfied in this case, since we have used the approximation $\bar{y} \ll 1$ in order to determine the parameters of the near flow field [2]. From (3.4) we have on the surface of the thin body ($\bar{y} = \delta Y(\bar{x})$):

$$Y_{\bar{x}} = \frac{1}{\bar{y}}\left[f_2(0)\exp(-\Lambda\bar{x}) + \int_0^{\bar{x}} f_2'(\bar{x}-t)\exp(-\Lambda t)\,dt\right]. \tag{3.5}$$

Obviously $f_2(0) = 0$ when $Y(0) = 0$, $Y_{\bar{x}}(0) \neq \infty$, i.e., the tip of the body located at $\bar{x} = 0$ is not blunt. Therefore the assumption (1.9) that $\Phi_{\xi}(0, \zeta) = 0$ is correct and we can write

$$\Phi_{\bar{x}\bar{y}} = -\frac{1}{\bar{y}}\int_0^{\bar{x}} f_2'(\eta)\exp(-\Lambda(\bar{x}-\eta))\,d\eta.$$

From (3.5) we then have

$$Y_{\bar{x}} = \frac{1}{\bar{y}}\int_0^{\bar{x}} f_2'(\eta)\exp(-\Lambda(\bar{x}-\eta))\,d\eta. \tag{3.6}$$

In place of the function $Y = Y(\bar{x})$ we use the cross-sectional area $S(\bar{x}) = \pi Y^2$ of the thin body. We note that $S'(0) = 2\pi Y(0)Y_{\bar{x}}(0) = 0$ since $Y(0) = 0$ and $Y_{\bar{x}}(0) \neq \infty$. From (3.6) we have (for $\bar{y} = \delta Y(\bar{x})$)

$$f_2'(\bar{x}) = \frac{\delta}{2\pi}\left(S'(\bar{x})\exp(\Lambda\bar{x})\right)_{\bar{x}}\exp(-\Lambda\bar{x}),$$

then

$$\Phi_{\bar{x}} = \frac{\delta}{2\pi}\left(S'(\bar{x})\ln\frac{2}{\sqrt{M_0^2-1}\,\bar{y}} + \int_0^{\bar{x}}\left(S'(\eta)\exp(\Lambda(\eta-\bar{x}))\right)_\eta \ln(\bar{x}-\eta)\,d\eta\right).$$

The drag force on a thin body of revolution is

$$\bar{D}_s = \frac{D_s}{\rho_0 u_0^2 L^2} = \delta^2\int_0^1 \frac{p-p_0}{\rho_0 u_0^2}\,dS = \delta^2\int_0^1 \frac{\gamma\bar{p}-1}{\gamma M_0^2}\,dS. \tag{3.7}$$
$$\gamma\bar{p}-1 = -\varepsilon\gamma p_{10} + \delta\gamma p_{01} - \varepsilon^2\gamma p_{20}.$$

For the calculations below, however, we will need to use the relation

$$M_0^2\left(\delta u_{01} + \frac{1}{2}(\delta v_{01})^2\right) + \delta p_{01} = 0.$$

Indeed

$$u_{01} = -\Phi_{\bar{x}\bar{x}},\quad v_{01} = -\Phi_{\bar{x}\bar{y}}.$$

In the limit $\bar{y} \to 0$ we have

$$\Phi_{\bar{x}\bar{x}} = \delta\ln\frac{2}{\sqrt{M_0^2-1}\,\bar{y}}\,\frac{S''(\bar{x})}{2\pi},\quad \Phi_{\bar{x}\bar{y}} = -\frac{\delta}{\bar{y}}\,\frac{S'(\bar{x})}{2\pi}.$$

On the surface of the thin body of revolution $\bar{y} = \delta Y(\bar{x}) = O(\delta)$; hence, $\Phi_{\bar{x}\bar{y}}$ is $O(1)$ and $\Phi_{\bar{x}\bar{x}}$ is $O(\delta\ln\delta)$. Therefore for values of δ of practical interest ($\delta = 0.1$, for example) the quantity $(\delta v_{01})^2$ is $O(\delta^2)$ and is of the same importance as δu_{01}, which is $O(\delta^2\ln\delta)$. For the drag force on the thin body we obtain from (3.7)

$$\bar{D}_s = -\delta^2\int_0^1(\varepsilon u_{10}+\varepsilon^2 u_{20})S'(\bar{x})\,d\bar{x} + \frac{\delta^4}{2\pi}\int_0^1\left[S''(\bar{x})\ln\frac{2}{\sqrt{M_0^2-1}\,\delta Y} + \right.$$
$$\left. + \left(\int_0^{\bar{x}}\left(S'(\eta)\exp(\Lambda(\eta-\bar{x}))\right)_\eta\ln(\bar{x}-\eta)\,d\eta\right)_{\bar{x}} - \pi(Y_{\bar{x}})^2\right]S'(\bar{x})\,d\bar{x}.$$

Here it is convenient to introduce the transformation

$$\int_0^1 S'(\bar{x})S''(\bar{x})\ln\frac{2}{\sqrt{M_0^2-1}\,\delta Y}\,d\bar{x} = \frac{1}{2}\ln\frac{2}{\sqrt{M_0^2-1}\,\delta Y(1)}(S'(1))^2 +$$
$$+\pi\int_0^1 S'(\bar{x})(Y_{\bar{x}})^2\,d\bar{x}.$$

This expression simplifies considerably if $S'(1) = 0$, which is the case when $Y(1) = 0$ (the contour of the body is closed at the aft end of the body) or when $Y_{\bar{x}}(1) = 0$ (the slope of the contour vanishes at the aft end). We assume that the condition $S'(1) = 0$ is satisfied for the body under consideration. After a transformation of the form

$$\int_0^1 S'(\bar{x})\left(\int_0^{\bar{x}} (S'(\eta)\exp(\Lambda(\eta-\bar{x})))_\eta \ln(\bar{x}-\eta)\,d\eta\right)_{\bar{x}} d\bar{x} =$$

$$= -\int_0^1 S''(\bar{x})\int_0^{\bar{x}} (S'(\eta)\exp(\Lambda(\eta-\bar{x})))_\eta \ln(\bar{x}-\eta)\,d\eta d\bar{x}$$

the formula for the drag force on the thin body takes the form

$$\bar{D}_s = -\delta^2\int_0^1 (\varepsilon u_{10} + \varepsilon^2 u_{20} + O(\varepsilon^3))\,S'(\bar{x})\,d\bar{x} - \tag{3.8}$$

$$-\frac{\delta^4}{2\pi}\int_0^1\left[S''(\bar{x})\int_0^{\bar{x}} (S'(\eta)\exp(\Lambda(\eta-\bar{x})))_\eta \ln(\bar{x}-\eta)\,d\eta + O(\varepsilon)\right] d\bar{x} + O(\delta^6).$$

When $\delta = o(\sqrt{\varepsilon})$ the leading term in (3.8) is

$$\bar{D}_s = -\varepsilon\delta^2\int_0^1 u_{10}(\bar{x})\,S'(\bar{x})\,d\bar{x}.$$

When $\delta = \sqrt{\varepsilon}$ the leading term in (3.8) can be written as

$$\bar{D}_s = -\delta^2\int_0^1\left(\varepsilon u_{10}S'(\bar{x}) + \delta^2\frac{S''(\bar{x})}{2\pi}\int_0^{\bar{x}} (S'(\eta)\exp(\Lambda(\eta-\bar{x})))_\eta \ln(\bar{x}-\eta)\,d\eta\right)d\bar{x}.$$

For $\varepsilon = o(\delta^2)$ the leading term in (3.8) is the second term. In the absence of relaxation ($\omega_0 \to 0$) (3.8) transforms to the Carman–Moore formula [3, 5]

$$\bar{D}_{s0} = -\frac{\delta^4}{2\pi}\int_0^1 S''(\bar{x})\int_0^{\bar{x}} S''(\eta)\ln(\bar{x}-\eta)\,d\eta d\bar{x}. \tag{3.9}$$

As in the case of a plane thin body, for small enough δ the first term in (3.8) is larger than the other terms in absolute value. It can be shown that

$$-\int_0^1 u_{10}S'(\bar{x})\,d\bar{x} = -u_{10}(1)\,S'(1) + \int_0^1 u_{10\bar{x}}S'(\bar{x})\,d\bar{x} = \int_0^1 u_{10\bar{x}}S'(\bar{x})\,d\bar{x} < 0.$$

Therefore there exists a value δ_{max} such that the thin body experiences negative drag when $\delta < \delta_{max}$ and positive drag when $\delta > \delta_{max}$. The value of δ_{max} is found from the condition $\bar{D}_s = 0$:

$$\delta_{max}^2 = \frac{2\pi\int_0^1 (\varepsilon u_{10} + \varepsilon^2 u_{20})\,S'(\bar{x})\,d\bar{x}}{\int_0^1 S''(\bar{x})\int_0^{\bar{x}} (S'(\eta)\exp(\Lambda(\eta-\bar{x})))_\eta \ln(\bar{x}-\eta)\,d\eta d\bar{x}}.$$

The coefficients σ and Λ given by (1.7) and (2.2) are approximately independent of M_0 when $M_0 > 3$. According to (1.5), ε is proportional to $1/M_0^2$ for $M_0 > 3$. Therefore δ_{max} behaves as $1/M_0$.

We next calculate $\bar{D}_s$ for a particular body of revolution using the flow parameters and constants characterizing the physical properties of the gas given in [10]. As in the plane case we assume that the generatrix of the body is $Y = 2\bar{x}(1 - \bar{x})$ and put $\delta = 0.1$.

TABLE 3

T_k, K	p, Pa	T, K	D_1	D_2	D_3	ε	$\bar{\omega}_0$	Λ
2000	101320	300	0,41	0,41	0,34	0,24	0,067	0,00001
2000	101320	1000	0,41	0,023	0,023	0,052	1914,0	60,0
3000	101320	430	0,41	0,41	—0,78	0,35	2,3	0,004

TABLE 4

T_k, K	p, Pa	T, K	D_1	D_2	D_3	ε	$\bar{\omega}_0$	Λ
2000	101320	300	0,41	0,41	0,23	0,27	16,0	0,004
2000	10132	1000	0,41	0,003	0,003	0,054	4834,0	176,0
2000	101320	250	0,41	0,41	—0,62	0,32	3,3	0,0002

The calculated results for molecular nitrogen and carbon monoxide are given in Tables 3 and 4, respectively, where D_1 is calculated from (3.9), D_2 and D_3 are calculated from (3.8) (D_2 for $\varepsilon = 0$). The D_i are multiplied by 10^{-3}. It was assumed that $M_0 = 2$.

The results obtained here in the linear theory show that vibrational relaxation can significantly change the drag force on a thin body in steady supersonic flow. In a nonexcited vibrationally relaxing gas pressure perturbations produced by the thin body damp out, which leads to a lower (higher) pressure at the forward (aft) part of the body than in the case of equilibrium. In the resultant pressure field the drag force on the thin body is smaller. In a vibrationally excited gas relaxation of the molecular excitations leads to growth of the pressure downstream, which results in a buoyancy force directed toward the incident flow. This force can lower the drag further and even reverse it (negative drag).

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